

COVARIANT LYAPUNOV EXPONENTS FOR THE MIXMASTER

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The dynamics of the Mixmaster Universe is analyzed in a covariant picture via Misner–Chitre-like variables for an ADM Hamiltonian approach. The system outcomes as isomorphic to a billiard on the Lobachevsky plane and Lyapunov exponents are calculated explicitly.

1. Introduction

One of the most outstanding problem discussed during the 1990s regarding the Mixmaster^{1,2} cosmology concerned the covariant nature of its chaoticity³. The subtleties arose in this debate because of the non-covariant nature of the standard chaos indicators when applied to General Relativity^{4,5,6}. The peculiarity of the Mixmaster model is due to the vanishing of its Hamiltonian and to the non-positive character of the corresponding kinetic term. Reliable indications about the chaos covariance appeared in later literature^{7,8,9}.

Here we show that the Mixmaster is isomorphic to a billiard ball on a Lobachevsky plane and that this picture is independent on the lapse-function choice; therefore we can apply the standard Lyapunov exponents to characterize the chaos covariance. We base our analysis over the existence of an asymptotic energy-like constant of motion which permits the definition of a standard Jacobi metric for the Arnowitt–Deser–Misner (ADM) reduced model⁹.

2. Covariant ADM Reduction

Starting from Misner variables α, β_+, β_- ¹⁰ we define the new Misner–Chitre-like ones $f(\tau), \xi, \theta$

$$\alpha = -e^{f(\tau)}\xi, \quad \beta_+ = e^{f(\tau)}\sqrt{\xi^2 - 1} \cos \theta, \quad \beta_- = e^{f(\tau)}\sqrt{\xi^2 - 1} \sin \theta, \quad (1)$$

with f denoting a generic functional form of τ , $1 \leq \xi < \infty$ and $0 \leq \theta < 2\pi$.

In terms of (1) the ADM reduction of the Mixmaster dynamics provides the action

$$\mathcal{S}_{\text{RED}} = \int \left(p_\xi d\xi + p_\theta d\theta - \sqrt{\varepsilon^2 + U} df \right), \quad (2)$$

where p_ξ and p_θ denote the conjugate momenta to ξ and θ , respectively,

$$\varepsilon^2 = (\xi^2 - 1) p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1} \quad (3)$$

and $U(\xi, \theta, \tau)$ can be modelled near the singularity ($\tau \rightarrow \infty$) by the infinite walls

$$U = \sum_i \Theta_\infty(H_i), \quad \Theta_\infty(x) = \begin{cases} 0 & x > 0 \\ \infty & x \leq 0; \end{cases} \quad (4)$$

the *anisotropy parameters* H_i ($\sum_i H_i = 1$) in the Misner–Chitre-like variables, read

$$H_{1,2} = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} \left(\cos \theta \pm \sqrt{3} \sin \theta \right), \quad H_3 = \frac{1}{3} + 2 \frac{\sqrt{\xi^2 - 1}}{3\xi} \cos \theta, \quad (5)$$

which do not depend on the (time) variable τ .

The domain Γ_H where U vanishes is dynamically closed and, within it, ε behaves as a constant of motion, i.e. $\frac{d\varepsilon}{d\tau} = \frac{\partial \varepsilon}{\partial \tau} = 0$, which implies $\varepsilon = E = \text{const.}$

The covariance of this picture is ensured because of the time-gauge relation

$$N(\tau) = \frac{12D}{E} e^{2f} \frac{df}{d\tau}, \quad (6)$$

in which $D = \exp(-3\xi e^{f(\tau)})$. The fixing of a specific time variable corresponds to choose a suitable function $f(\tau)$.

3. Lyapunov Exponents

In Γ_H the dynamics of the Mixmaster is summarized by the variational principle

$$\delta \int (p_\xi d\xi + p_\theta d\theta) = 0; \quad (7)$$

this picture can be restated in terms of a billiard geodesic flow on the Lobachevsky plane described by the line-element⁹

$$dl^2 = E^2 \left[\frac{d\xi^2}{\xi^2 - 1} + (\xi^2 - 1) d\theta^2 \right]. \quad (8)$$

Such a space has a constant negative curvature (the Ricci scalar is equal to $-2/E^2$) and the dynamical instability is studied via the geodesic deviation equation as projected over the Fermi basis

$$v^i = \left(\frac{1}{E} \sqrt{\xi^2 - 1} \cos \phi, \frac{1}{E} \frac{\sin \phi}{\sqrt{\xi^2 - 1}} \right) \quad (9a)$$

$$w^i = \left(-\frac{1}{E} \sqrt{\xi^2 - 1} \sin \phi, \frac{1}{E} \frac{\cos \phi}{\sqrt{\xi^2 - 1}} \right), \quad (9b)$$

where $\phi(\tau)$ lies in the interval $[0, 2\pi)$; the vector v^i is the geodesic field while w^i is parallelly transported along it.

Projecting the geodesic deviation equation along the vector w^i (its component along the geodesic field v^i does not provide any physical information about the system instability), the corresponding connecting vector (tetradic) component Z satisfies the equivalent equation

$$\frac{d^2 Z}{ds^2} = \frac{Z}{E^2}. \quad (10)$$

Expression (10), as a projection on the tetradic basis, is a scalar and therefore completely independent of the choice of the variables. Its general solution reads

$$Z(s) = c_1 e^{\frac{s}{E}} + c_2 e^{-\frac{s}{E}}, \quad c_{1,2} = \text{const.}, \quad (11)$$

and the invariant Lyapunov exponent defined by

$$\lambda_v = \sup \lim_{s \rightarrow \infty} \frac{\ln \left(Z^2 + \left(\frac{dZ}{ds} \right)^2 \right)}{2s}, \quad (12)$$

takes the value

$$\lambda_v = \frac{1}{E} > 0. \quad (13)$$

When the point-universe bounces against the potential walls, it is reflected from a geodesic to another one thus making each of them unstable. Though up to the limit of our potential wall approximation, this result shows without any ambiguity that, independently of the choice of the temporal gauge, the Mixmaster dynamics is isomorphic to a well described chaotic system.

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